Propositional Resolution

Second Part: Algorithms

Benjamin Wack

Université Grenoble Alpes

February 1st, 2017
Proof by resolution of our running example

- (H1) : $p \Rightarrow \neg j \equiv \neg p \lor \neg j$
- (H2) : $\neg p \Rightarrow j \equiv p \lor j$
- (H3) : $j \Rightarrow m \equiv \neg j \lor m$
- ($\neg C$): $\neg m \land \neg p$
Proof by resolution of our running example

- (H1) : $p \Rightarrow \neg j \equiv \neg p \lor \neg j$
- (H2) : $\neg p \Rightarrow j \equiv p \lor j$
- (H3) : $j \Rightarrow m \equiv \neg j \lor m$
- (¬ C) : $\neg m \land \neg p$

Clauses: $\{\neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p\}$
Proof by resolution of our running example

- (H1): \( p \Rightarrow \neg j \equiv \neg p \lor \neg j \)
- (H2): \( \neg p \Rightarrow j \equiv p \lor j \)
- (H3): \( j \Rightarrow m \equiv \neg j \lor m \)
- (\neg C): \( \neg m \land \neg p \)

Clauses: \( \{ \neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p \} \)

\[
\begin{array}{c}
p \lor j \\
\hline
\neg j \lor m \\
\hline
\neg m \\
\hline
\neg m \\
\hline
\neg p \\
\hline
p \\
\hline
\bot
\end{array}
\]
Proof by resolution of our running example

▶ (H1) : \( p \Rightarrow \neg j \equiv \neg p \lor \neg j \)

▶ (H2) : \( \neg p \Rightarrow j \equiv p \lor j \)

▶ (H3) : \( j \Rightarrow m \equiv \neg j \lor m \)

▶ (\( \neg C \)) : \( \neg m \land \neg p \)

Clauses: \( \{ \neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p \} \)

\[
\begin{array}{c}
p \lor j & \neg j \lor m & \neg m \\
\hline
p \lor m & & \neg p
\end{array}
\]

\[
\begin{array}{c}
p \lor j & \neg p & \neg j \lor m \\
\hline
j & m & \neg m
\end{array}
\]

OR...

OR...
Last course

- Boolean Algebra
- Boolean functions
- Resolution

(1) $A \vdash B$

$B$ is deduced from $A$: there is a proof by resolution of $B$ starting from $A$. 
Last course

- Boolean Algebra
- Boolean functions
- Resolution

(1) \( A \vdash B \)

\( B \) is deduced from \( A \): there is a proof by resolution of \( B \) starting from \( A \).

(2) \( A \models B \)

\( B \) is a consequence of \( A \): every model of \( A \) is also a model of \( B \).
Last course

- Boolean Algebra
- Boolean functions
- Resolution

(1) $A \vdash B$

$B$ is deduced from $A$: there is a proof by resolution of $B$ starting from $A$.

(2) $A \models B$

$B$ is a consequence of $A$: every model of $A$ is also a model of $B$.

Today: Correctness

(1) $\Rightarrow$ (2)
Last course

- Boolean Algebra
- Boolean functions
- Resolution

1. \( A \vdash B \)
   
   \( B \) is deduced from \( A \): there is a proof by resolution of \( B \) starting from \( A \).

2. \( A \models B \)
   
   \( B \) is a consequence of \( A \): every model of \( A \) is also a model of \( B \).

Today: Correctness

1. \( \Rightarrow \) 2

Today: Completeness

2. \( \Rightarrow \) 1
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Definition

The correctness of a deductive system states that all proofs obtained in this system “prove only true statements”.
Correctness of the resolution rule

**Theorem 2.1.15**

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

**Proof.**

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A$, $L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $v$ be an assignment such that $[A]_v = 1$ and $[B]_v = 1$: let us show that $[C]_v = 1$.

- Suppose that $[L]_v = 1$. Therefore $[L^c]_v = 0$. Since $[B]_v = 1$, $v$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_v = 1$.

- Suppose that $[L^c]_v = 1$. Therefore $[L]_v = 0$. Since $[A]_v = 1$, $v$ is a model of $(A - \{L\})$. Hence $[C]_v = 1$.

Since every truth assignment is either model of $L$ or $L^c$, $v$ is a model of $C$. 


Correctness of the resolution rule

Theorem 2.1.15
If \( C \) is a resolvent of \( A \) and \( B \) then \( A, B \models C \).

Proof.
If \( C \) is a resolvent of \( A \) and \( B \), then there is a literal \( L \) such that \( L \in A, L^c \in B \), and \( C = (A - \{L\}) \cup (B - \{L^c\}) \).
Correctness of the resolution rule

Theorem 2.1.15
If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.
If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.
Let $\nu$ be an assignment such that $[A]_{\nu} = 1$ and $[B]_{\nu} = 1$: let us show that $[C]_{\nu} = 1$. 

Correctness of the resolution rule

**Theorem 2.1.15**

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

**Proof.**

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $\nu$ be an assignment such that $[A]_\nu = 1$ and $[B]_\nu = 1$: let us show that $[C]_\nu = 1$.

- Suppose that $[L]_\nu = 1$.
- Suppose that $[L^c]_\nu = 1$. 
Correctness of the resolution rule

Theorem 2.1.15

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $\nu$ be an assignment such that $[A]_\nu = 1$ and $[B]_\nu = 1$: let us show that $[C]_\nu = 1$.

- Suppose that $[L]_\nu = 1$. Therefore $[L^c]_\nu = 0$.
  Since $[B]_\nu = 1$, $\nu$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_\nu = 1$.

- Suppose that $[L^c]_\nu = 1$. 

\[ \square \]
Correctness of the resolution rule

**Theorem 2.1.15**

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

**Proof.**

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A$, $L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $v$ be an assignment such that $[A]_v = 1$ and $[B]_v = 1$: let us show that $[C]_v = 1$.

- Suppose that $[L]_v = 1$. Therefore $[L^c]_v = 0$.
  Since $[B]_v = 1$, $v$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_v = 1$.

- Suppose that $[L^c]_v = 1$. Therefore $[L]_v = 0$.
  Since $[A]_v = 1$, $v$ is a model of $(A - \{L\})$. Hence $[C]_v = 1$. 

Correctness of the resolution rule

Theorem 2.1.15

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $v$ be an assignment such that $[A]_v = 1$ and $[B]_v = 1$: let us show that $[C]_v = 1$.

- Suppose that $[L]_v = 1$. Therefore $[L^c]_v = 0$.
  Since $[B]_v = 1$, $v$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_v = 1$.

- Suppose that $[L^c]_v = 1$. Therefore $[L]_v = 0$.
  Since $[A]_v = 1$, $v$ is a model of $(A - \{L\})$. Hence $[C]_v = 1$.

Since every truth assignment is either model of $L$ or $L^c$, $v$ is a model of $C$. 

□
Correctness of deduction

Theorem 2.1.16

Let $\Gamma$ be a set of clauses and $C$ a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

Proof.

Suppose that there is a proof $P$ of $C$ starting from $\Gamma$.
Suppose that for any proof of $\Gamma \vdash D$ shorter than $P$, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. $C$ is a member of $\Gamma$, in this case $\Gamma \models C$.
2. $\Gamma \vdash A$, $\Gamma \vdash B$ (with a shorter proof) and $A \land B \vdash C$
   By induction hypothesis: $\Gamma \models A$ and $\Gamma \models B$.
   By correctness of the resolution rule: $A \land B \models C$.
   Hence $\Gamma \models C$.  

B. Wack et al (UGA)
Correctness of deduction

**Theorem 2.1.16**

Let $\Gamma$ be a set of clauses and $C$ a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

**Proof.**

Suppose that there is a proof $P$ of $C$ starting from $\Gamma$.
Suppose that for any proof of $\Gamma \vdash D$ shorter than $P$, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. $C$ is a member of $\Gamma$, in this case $\Gamma \models C$. 
Correctness of deduction

Theorem 2.1.16

Let $\Gamma$ be a set of clauses and $C$ a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

Proof.

Suppose that there is a proof $P$ of $C$ starting from $\Gamma$.
Suppose that for any proof of $\Gamma \vdash D$ shorter than $P$, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. $C$ is a member of $\Gamma$, in this case $\Gamma \models C$.
2. $\Gamma \vdash A$, $\Gamma \vdash B$ (with a shorter proof) and

$$
\begin{array}{c}
A \quad B \\
\hline \\
C
\end{array}
$$

By induction hypothesis: $\Gamma \models A$ and $\Gamma \models B$.
By correctness of the resolution rule: $A, B \models C$. Hence $\Gamma \models C$. 

□
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Definition

Completeness for the refutation is the property: If $\Gamma \models \bot$ then $\Gamma \vdash \bot$.

We prove this result for a finite $\Gamma$. 
Definition 2.1.18

Let $\Gamma$ be a set of clauses and $L$ a literal.

$\Gamma[L := 1]$ is obtained by:
- deleting the clauses containing $L$
- removing $L^c$ from the other clauses.

$\Gamma[L := 0]$ is similarly defined by switching the roles of $L$ and $L^c$.

Remark: the number of variables in $\Gamma$ has been decreased.
Examples

Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q, \overline{q} + r, p + q, p + r$. We have:

- $\Gamma[p := 1] =$
- $\Gamma[p := 0] =$

Notice that:

- $(1 + q)(q + r)(1 + q)(1 + r) \equiv q(q + r) = \Gamma[p := 1]$
- $(0 + q)(q + r)(0 + q)(0 + r) \equiv (q + r) = \Gamma[p := 0]$
Examples

Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] =$
  
  $\{q, \overline{q} + r\}$.

- $\Gamma[p := 0] =$
Examples

Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] = \{q, \overline{q} + r\}$.
- $\Gamma[p := 0] = \{\overline{q} + r, q, r\}$. 

Notice that:

- $$(\overline{1} + q)(q + r)(\overline{1} + q)(\overline{1} + r) \equiv q(q + r) = \Gamma[p := 1]$$
- $$(\overline{0} + q)(q + r)(\overline{0} + q)(\overline{0} + r) \equiv (q + r)qr = \Gamma[p := 0]$$
Examples

Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] =$
  \{ $q$, $\overline{q} + r$ \}.

- $\Gamma[p := 0] =$
  \{ $\overline{q} + r$, $q$, $r$ \}.

Notice that:

- $(\overline{1} + q)(\overline{q} + r)(1 + q)(1 + r) \equiv q(\overline{q} + r) = \Gamma[p := 1].$

- $(\overline{0} + q)(\overline{q} + r)(0 + q)(0 + r) \equiv (\overline{q} + r)qr = \Gamma[p := 0].$
<table>
<thead>
<tr>
<th>Property 2.1.21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.</td>
</tr>
</tbody>
</table>

**Proof.**
Property of $\Gamma[L := \ldots]$ 

Property 2.1.21

$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

$\Rightarrow$ If $\nu$ is a model of $\Gamma$ then $\nu$ is a model of either $\Gamma[L := 0]$ (if $[L]\nu = 0$) or $\Gamma[L := 1]$ (if $[L]\nu = 1$).
Property of $\Gamma[L := \ldots]$ 

Property 2.1.21

$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

$\Rightarrow$ If $v$ is a model of $\Gamma$ then $v$ is a model of either $\Gamma[L := 0]$ (if $[L]_v = 0$) or $\Gamma[L := 1]$ (if $[L]_v = 1$)

$\Leftarrow$ If $v$ is a model of $\Gamma[L := i]$ then we can build a model of $\Gamma$ (by taking $[L]_v = i$)
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal.
If $\Gamma[L := 1] \not\vdash C$ then $\Gamma \not\vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:

- If $C$ is a resolvent of $A$ and $B$:
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal.
If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:
  - either $C$ was in $\Gamma$, thus $\Gamma \vdash C$
  - or $C$ was obtained by removing a $L^c$, thus $\Gamma \vdash C + L^c$

- If $C$ is a resolvent of $A$ and $B$: 

\[\square\]
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal. If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$: 
  - either $C$ was in $\Gamma$, thus $\Gamma \vdash C$
  - or $C$ was obtained by removing a $L^c$, thus $\Gamma \vdash C + L^c$

- If $C$ is a resolvent of $A$ and $B$:
  - either $\Gamma \vdash A$ and $\Gamma \vdash B$, hence $\Gamma \vdash C$
  - or $L^c$ has to be put back into $A$ or $B$, thus into $C$ too
Completeness of propositional resolution

Theorem 2.1.24

Let $\Gamma$ be a finite set of clauses. If $\Gamma$ is unsatisfiable then $\Gamma \vdash \bot$.

Proof

By induction on the number of variables in $\Gamma$.

★ Base case: $\Gamma$ has no variable, so $\Gamma = \emptyset$ (impossible, it’s valid) or $\Gamma = \{\bot\}$.

★ Inductive step: either we prove directly that $\Gamma \vdash \bot$, or that $\Gamma \vdash x$ and $\Gamma \vdash \overline{x}$.

Corollary 2.1.25

$\Gamma$ is unsatisfiable if and only if $\Gamma \vdash \bot$. 
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?

- **Complete strategy**
  Construction of ALL the deductible clauses (resolvents) from $\Gamma$
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?

- **Complete strategy**
  Construction of ALL the deductible clauses (resolvents) from $\Gamma$

- **The Davis-Putnam-Logemann-Loveland Algorithm**
  “Intelligent” traversal of the possible assignments of $\Gamma$
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?

- **Complete strategy**
  Construction of ALL the deductible clauses (resolvents) from $\Gamma$

- **The Davis-Putnam-Logemann-Loveland Algorithm**
  “Intelligent” traversal of the possible assignments of $\Gamma$

**Remark**

*Exponential* solutions in time in the worst case.
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Exponential complexity

Remember that two clauses having the same set of literals are equal.

If $\Gamma$ uses $n$, then we have at most $2^n$ distinct clauses deduced from $\Gamma$. 
Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.
Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.

How to proceed with reduction?

Remove the valid clauses and the clauses containing another clause of the set.
Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.

How to proceed with reduction?

Remove the valid clauses and the clauses containing another clause of the set.

Example 2.1.27

The reduction of the set of clauses \{p + q + \overline{p}, p + r, p + r + \overline{s}, r + q\} gives the reduced set:
Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.

How to proceed with reduction?

Remove the valid clauses and the clauses containing another clause of the set.

Example 2.1.27

The reduction of the set of clauses \( \{p + q + \overline{p}, p + r, p + r + \overline{s}, r + q\} \)
gives the reduced set:

\( \{p + q + \overline{p}, p + r, p + r + \overline{s}, r + q\} \).
Justification

Property 2.1.28

A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$. 
Justification

Property 2.1.28

A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$.

Proof.

- Removing valid clauses: $x.1 \equiv x$
Justification

Property 2.1.28

A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$.

Proof.

- Removing valid clauses: $x.1 \equiv x$
- Removing a clause including another clause: $x(x + y) \equiv x$
Result of the algorithm: minimum deduction clauses

Definition 2.1.29

A minimum clause for the deduction from $\Gamma$ is:
- a non-valid clause
- deduced from $\Gamma$
- and containing no other clause deduced from $\Gamma$. 
Result of the algorithm: minimum deduction clauses

Definition 2.1.29

A minimum clause for the deduction from $\Gamma$ is:

- a non-valid clause
- deduced from $\Gamma$
- and containing no other clause deduced from $\Gamma$.

Example 2.1.30

$\Gamma = \{ a + \overline{b}, b + c + d \}$

- The clause $a + c + d$ is a minimum clause for deduction.
**Result of the algorithm: minimum deduction clauses**

**Definition 2.1.29**

A minimum clause for the deduction from $\Gamma$ is:

- a non-valid clause
- deduced from $\Gamma$
- and containing no other clause deduced from $\Gamma$.

**Example 2.1.30**

$\Gamma = \{a + \overline{b}, b + c + d\}$

- The clause $a + c + d$ is a minimum clause for deduction.
- But if we add $\overline{a} + c$ to $\Gamma$, then $a + c + d$ is not minimal anymore (since we can now deduce $c + d$).
Property 2.1.31

Let $\Theta$ be the set of minimum deduction clauses for the set of clauses $\Gamma$. $\Gamma$ is unsatisfiable if and only if $\bot \in \Theta$. 
Property 2.1.31

Let $\Theta$ be the set of minimum deduction clauses for the set of clauses $\Gamma$. $\Gamma$ is unsatisfiable if and only if $\bot \in \Theta$.

Proof.

- Suppose $\bot \in \Theta$, then $\Gamma \vdash \bot$, hence by resolution correctness, $\Gamma$ is unsatisfiable.
Property

Property 2.1.31

Let $\Theta$ be the set of minimum deduction clauses for the set of clauses $\Gamma$. $\Gamma$ is unsatisfiable if and only if $\bot \in \Theta$.

Proof.

- Suppose $\bot \in \Theta$, then $\Gamma \vdash \bot$, hence by resolution correctness, $\Gamma$ is unsatisfiable.
- Suppose $\Gamma$ is unsatisfiable, by resolution completeness, $\Gamma \vdash \bot$. Consequently $\bot$ is a minimum clause for deduction from $\Gamma$, therefore $\bot \in \Theta$. 

□
Principle of the algorithm: Build all the clauses deduced from $\Gamma$ 

Following the height of the proof trees.

Algorithm

For any integer $i$
While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most $i$. 
Principle of the algorithm: Build all the clauses deduced from $\Gamma$

Following the height of the proof trees.

Algorithm

For any integer $i$
While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most $i$.

In practice:
Maintain two sequences of the sets of clauses, $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$
Two sequences of sets of clauses

$\Delta_i$ are the new useful clauses

Clauses deduced from $\Gamma$ by a proof of height $i$, after removal of:

- valid clauses
- clauses including another clause whose proof has height $< i$.  

$\Delta_0$ is obtained by reducing $\Gamma$. 
Two sequences of sets of clauses

$\Delta_i$ are the new useful clauses

Clauses deduced from $\Gamma$ by a proof of height $i$, after removal of:

- valid clauses
- clauses including another clause whose proof has height $< i$.

$\Delta_0$ is obtained by reducing $\Gamma$.

$\Theta_i$ are the old clauses still useful

Clauses deduced from $\Gamma$ by a proof of height $< i$ after removal of:

- valid clauses
- clauses including another clause whose proof has height $\leq i$.

$\Theta_0$ is the empty set.
Construction of the sequences $\Delta_i(i \geq 0)$ and $\Theta_i(i \geq 0)$

$\Delta_{i+1}$

- Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause from $\Delta_i \cup \Theta_i$
Construction of the sequences $\Delta_i(i \geq 0)$ and $\Theta_i(i \geq 0)$

$\Delta_{i+1}$

- Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause from $\Delta_i \cup \Theta_i$

$\Theta_{i+1}$

Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from $\Delta_{i+1}$. 
Construction of the sequences $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$

<table>
<thead>
<tr>
<th>$\Delta_{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$</td>
</tr>
<tr>
<td>▶ Reduce this set</td>
</tr>
<tr>
<td>▶ Remove the new resolvents including a clause from $\Delta_i \cup \Theta_i$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Theta_{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from $\Delta_{i+1}$.</td>
</tr>
</tbody>
</table>

When $\Delta_k = \emptyset$, stop the construction:

| |
| ▶ $k - 1$ is then the maximum height of a proof |
| ▶ $\Theta_k$ is the reduced set of the clauses deduced from $\Gamma$ |
Propositional Resolution
Complete strategy

Exemple 2.2.1

Soit $\Gamma = \{a + b + \overline{a}, \ a + b, \ a + b + c, \ a + \overline{b}, \ \overline{a} + b, \ \overline{a} + \overline{b}\}$

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$.
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit $\Gamma = \{a + b + \overline{a}, a + b, a + b + c, a + \overline{b}, \overline{a} + b, \overline{a} + \overline{b}\}$

$\begin{array}{|c|c|c|c|}
\hline
i & \Delta_i & \Theta_i & \Delta_i \cup \Theta_i & \text{Résolvants de } \Delta_i \text{ et } \Delta_i \cup \Theta_i \\
\hline
0 & & & & \\
\hline
\end{array}$

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 

Exemple 2.2.1

Soit $\Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b,$</td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$.
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b, a + b + c, a + \bar{b}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit \( \Gamma = \{ a + b + \overline{a}, a + b, a + b + c, a + \overline{b}, \overline{a} + b, \overline{a} + \overline{b} \} \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \Delta_i )</th>
<th>( \Theta_i )</th>
<th>( \Delta_i \cup \Theta_i )</th>
<th>Résolvants de ( \Delta_i ) et ( \Delta_i \cup \Theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a + b + \overline{a}, a + b, a + b + c, a + \overline{b}, \overline{a} + b, \overline{a} + \overline{b} )</td>
<td>( \emptyset )</td>
<td>( a + b, a + b, \overline{a} + b, \overline{a} + \overline{b} )</td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)
- \( \Theta_{i+1} = \)
  - Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
**Exemple 2.2.1**

Soit \( \Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \} \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \Delta_i )</th>
<th>( \Theta_i )</th>
<th>( \Delta_i \cup \Theta_i )</th>
<th>Résolvants de ( \Delta_i ) et ( \Delta_i \cup \Theta_i )</th>
</tr>
</thead>
</table>
| 0 | \( a + b + \bar{a}, a + b, a + b + c, a + \bar{b} \)
|   | \( \bar{a} + b, \bar{a} + \bar{b} \) | \( \emptyset \) | \( a + b, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \) | \( a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a} \) |

Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)

- \( \Theta_{i+1} = \)
  Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Exemple 2.2.1

Soit $\Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
</table>
| 0   | $a + b + \bar{a}, a + b$  
     | $a + b + c, a + \bar{b}$  
     | $\bar{a} + b, \bar{a} + \bar{b}$ | $\emptyset$ | $a + b, a + \bar{b}$,  
     |                                           | $\bar{a} + b, \bar{a} + \bar{b}$ | $a, b, b + \bar{b}$,  
     |                                           |                                           | $a + \bar{a}, \bar{b}, \bar{a}$ |
| 1   | $\bar{a} + b, \bar{a} + \bar{b}$ |                                           |                                           |                                           |

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit \( \Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \} \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \Delta_i )</th>
<th>( \Theta_i )</th>
<th>( \Delta_i \cup \Theta_i )</th>
<th>Résolvants de ( \Delta_i ) et ( \Delta_i \cup \Theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a + b + \bar{a}, a + b \ a + b + c, a + \bar{b} \ \bar{a} + b, \bar{a} + \bar{b} )</td>
<td>( \emptyset )</td>
<td>( a + b, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} )</td>
<td>( a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a} )</td>
</tr>
<tr>
<td>1</td>
<td>( a, b, b, \bar{a} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)
- \( \Theta_{i+1} = \)
  Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>$0$</td>
<td>$a + b, a + \bar{b}$, $\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>$\emptyset$</td>
<td>$a + b, a + \bar{b}$, $\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}$, $a + \bar{a}, b, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>$\emptyset$</td>
<td>$a, b, b, \bar{a}$</td>
<td>$a, b, b, \bar{a}$</td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$
- $\Theta_{i+1} =$
  - Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>0</td>
<td>$a + b, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, b, \bar{a}$</td>
<td>⊥</td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit $\Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}$</td>
<td>0</td>
<td>$a + b, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>2</td>
<td>$\Theta_{i+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  - Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$.
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>0</td>
<td>$a + b, a + \bar{b}$, $\bar{a} + \bar{b}, \bar{a} + b$</td>
<td>$a, b, b + \bar{b}$, $a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>2</td>
<td>$\bot$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

1. $\Delta_{i+1} =$
   - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
   - Reduce this set
   - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$
2. $\Theta_{i+1} =$
   - Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$.
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b, a + b + c, a + \bar{b}$</td>
<td>0</td>
<td>$a + b, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}, a + \bar{a}, b, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$\perp$</td>
</tr>
<tr>
<td>2</td>
<td>$\perp$</td>
<td>0</td>
<td>$a, b, b, \bar{a}$</td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 

Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$.
Exemple 2.2.1

Soit $\Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>0</td>
<td>$a + b, a + \bar{b}$</td>
<td>$a, b, b + \bar{b}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a + \bar{a}, b, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, b, \bar{a}$</td>
<td>$\perp$</td>
</tr>
<tr>
<td>2</td>
<td>$\perp$</td>
<td>0</td>
<td>$\perp$</td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$
- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Exemple 2.2.1

Soit \( \Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \} \)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\Delta_i)</th>
<th>(\Theta_i)</th>
<th>(\Delta_i \cup \Theta_i)</th>
<th>Résolvants de (\Delta_i) et (\Delta_i \cup \Theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a + b + \bar{a}, a + b)</td>
<td>0</td>
<td>(a + b, a + \bar{b}), (\bar{a} + b, \bar{a} + \bar{b})</td>
<td>(a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a})</td>
</tr>
<tr>
<td></td>
<td>(a + b + c, a + \bar{b})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\bar{a} + b, \bar{a} + \bar{b})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(a, b, b, \bar{a})</td>
<td>0</td>
<td>(a, b, b, \bar{a})</td>
<td>(\bot)</td>
</tr>
<tr>
<td>2</td>
<td>(\bot)</td>
<td>0</td>
<td>(\bot)</td>
<td>(\emptyset)</td>
</tr>
</tbody>
</table>

Rappel :

1. \(\Delta_{i+1} = \)
   - Compute all the resolvents of \(\Delta_i\) and \(\Delta_i \cup \Theta_i\)
   - Reduce this set
   - Remove the new resolvents which include a clause from \(\Delta_i \cup \Theta_i\)

2. \(\Theta_{i+1} = \)
   - Remove from \(\Delta_i \cup \Theta_i\) the clauses which include a clause of \(\Delta_{i+1}\).
**Exemple 2.2.1**

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>0</td>
<td>$a + b, a + b,$ $\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + b,$ $a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, b, \bar{a}$</td>
<td>0</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$\perp$</td>
</tr>
<tr>
<td>2</td>
<td>$\perp$</td>
<td>0</td>
<td>$\perp$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

1. $\Delta_{i+1} =$
   - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
   - Reduce this set
   - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

2. $\Theta_{i+1} =$
   - Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$.
Exemple 2.2.1

Soit $\Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
</table>
| 0   | $a + b + \bar{a}, a + b$  
    |             | $\emptyset$ | $a + b, a + b, \bar{a} + b, \bar{a} + \bar{b}$ | $a, b, b + b, a + \bar{a}, \bar{b}, \bar{a}$ |
|     | $a + b + c, a + \bar{b}$  
    |             |             |                             |                                                  |
|     | $\bar{a} + b, \bar{a} + \bar{b}$ |             |                             |                                                  |
| 1   | $a, b, b, \bar{a}$ | $\emptyset$ | $a, b, \bar{b}, \bar{a}$ | $\bot$                                      |
| 2   | $\bot$        | $\emptyset$ | $\bot$                     | $\emptyset$                                   |
| 3   | $\emptyset$   | $\bot$     | $\bot$                     | $\emptyset$                                   |

Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  - Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
The proof we built

1. $a + b$
2. $a + \overline{b}$
3. $\overline{a} + b$
4. $\overline{a} + \overline{b}$
5. $a$  \hspace{1cm} \text{resolvent of 1 and 2}
6. $b$  \hspace{1cm} \text{resolvent of 1 and 3}
7. $\overline{b}$  \hspace{1cm} \text{resolvent of 2 and 4}
8. $\overline{a}$  \hspace{1cm} \text{resolvent of 3 and 4}
9. $\perp$  \hspace{1cm} \text{resolvent of 5 and 8}
### The proof we built

<table>
<thead>
<tr>
<th></th>
<th>Expression</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a + b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$a + \overline{b}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\overline{a} + b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\overline{a} + \overline{b}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$a$</td>
<td>resolvent of 1 and 2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$b$</td>
<td>resolvent of 1 and 3</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\overline{b}$</td>
<td>resolvent of 2 and 4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\overline{a}$</td>
<td>resolvent of 3 and 4</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\perp$</td>
<td>resolvent of 5 and 8</td>
<td></td>
</tr>
</tbody>
</table>
Example 2.2.2

\[ \{ a, c, \overline{a} + \overline{b}, \overline{c} + e \} \]

Rappel :

1. \( \Delta_{i+1} = \)
   - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
   - Reduce this set
   - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)

2. \( \Theta_{i+1} = \)
   - Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Example 2.2.2

\[ \{ a, c, \overline{a} + \overline{b}, \overline{c} + e\} \]

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\Delta_i)</th>
<th>(\Theta_i)</th>
<th>(\Delta_i \cup \Theta_i)</th>
<th>Rés. de (\Delta_i) et (\Delta_i \cup \Theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(\emptyset)</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(\overline{b}, e)</td>
</tr>
</tbody>
</table>
Example 2.2.2

\[ \{ a, c, \overline{a} + \overline{b}, \overline{c} + e \} \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \Delta_i )</th>
<th>( \Theta_i )</th>
<th>( \Delta_i \cup \Theta_i )</th>
<th>Rés. de ( \Delta_i ) et ( \Delta_i \cup \Theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a, c, \overline{a} + \overline{b}, \overline{c} + e )</td>
<td>( \emptyset )</td>
<td>( a, c, \overline{a} + \overline{b}, \overline{c} + e )</td>
<td>( \overline{b}, e )</td>
</tr>
<tr>
<td>1</td>
<td>( \overline{b}, e )</td>
<td>( a, c )</td>
<td>( \overline{b}, e, a, c )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)

- \( \Theta_{i+1} = \)
  Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Example 2.2.2

\{a, c, \overline{a} + \overline{b}, \overline{c} + e\}

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\Delta_i)</th>
<th>(\Theta_i)</th>
<th>(\Delta_i \cup \Theta_i)</th>
<th>Rés. de (\Delta_i) et (\Delta_i \cup \Theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(\emptyset)</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(b, e)</td>
</tr>
<tr>
<td>1</td>
<td>(\overline{b}, e)</td>
<td>(a, c)</td>
<td>(\overline{b}, e, a, c)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>2</td>
<td>(\emptyset)</td>
<td>(\overline{b}, e, a, c)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

- \(\Delta_{i+1} = \)
  - Compute all the resolvents of \(\Delta_i\) and \(\Delta_i \cup \Theta_i\)
  - Reduce this set
  - Remove the new resolvents which include a clause from \(\Delta_i \cup \Theta_i\)

- \(\Theta_{i+1} = \)
  - Remove from \(\Delta_i \cup \Theta_i\) the clauses which include a clause of \(\Delta_{i+1}\)
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_i(i \geq 0)$ contains only clauses deduced from $\Gamma$.
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_{i(i\geq 0)}$ contains only clauses deduced from $\Gamma$

**Property 2.2.3**

Let $i \leq k$. Any clause of $\bigcup_{j\leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.

(by construction of $\Theta_i$)

**Property 2.2.4**

For all $i \leq k$, the sets $\Delta_i$ are mutually disjoint.

(by construction of $\Delta_i$)
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_i(i \geq 0)$ contains only clauses deduced from $\Gamma$

**Property 2.2.3**

Let $i \leq k$. Any clause of $\bigcup_{j \leq i} \Delta_j$ contains a clause of $\Delta_i \cup \Theta_i$.
(by construction of $\Theta_i$)

**Property 2.2.4**

For all $i \leq k$, the sets $\Delta_i$ are mutually disjoint.
(by construction of $\Delta_i$)

$\Delta_i(i \geq 0)$ are mutually disjoint

Hence there are at most $2^n + 1$ sets, therefore $k \leq 2^n + 1$
Interpretation

When the algorithm terminates:

if $\bot \in \Theta_k : \Gamma$ is **unsatisfiable**

if $\bot \notin \Theta_k : \Gamma$ is **satisfiable**, but what does $\Theta_k$ represent?
Interpretation

When the algorithm terminates:

if \( \bot \in \Theta_k \): \( \Gamma \) is unsatisfiable

if \( \bot \notin \Theta_k \): \( \Gamma \) is satisfiable, but what does \( \Theta_k \) represent?

Definition 2.1.32

A minimum clause for consequence from \( \Gamma \) is:

- a non valid clause
- which is a consequence of \( \Gamma \)
- not containing any other clause that is a consequence of \( \Gamma \).

Theorem 2.1.35

A clause is minimal for deduction IFF it is minimal for consequence.

Hence \( \Theta_k \) gives the minimum consequence clauses for \( \Gamma \).
Example 2.1.33

Let $\Gamma = \{a + d, \overline{a} + b, \overline{b} + c\}$. The clause $d + c$ is minimum for the consequence of $\Gamma$. 
Example 2.1.33

Let \( \Gamma = \{ a + d, \overline{a} + b, \overline{b} + c \} \).

The clause \( d + c \) is minimum for the consequence of \( \Gamma \).

Consequence: \( d + c \) is a consequence of \( \Gamma \) since in every model of \( \Gamma \), either \( d \) is true or \( c \) is true.
Example 2.1.33

Let $\Gamma = \{a + d, \overline{a} + b, \overline{b} + c\}$.

The clause $d + c$ is minimum for the consequence of $\Gamma$.

Consequence: $d + c$ is a consequence of $\Gamma$

since in every model of $\Gamma$, either $d$ is true or $c$ is true.

Minimality: The clauses contained in $c + d$ are $c$ and $d$.

$a = 0, d = 1, c = 0, b = 0$ is a model of $\Gamma$ but not of $c$.

$a = 1, d = 0, c = 1, b = 1$ is a model of $\Gamma$ but not of $d$. 
Result of the algorithm

- $\Theta_k =$ set of minimum deduction clauses.
- $\Gamma$ and $\Theta_k$ are equivalent.
Result of the algorithm

- $\Theta_k =$ set of minimum deduction clauses.
- $\Gamma$ and $\Theta_k$ are equivalent.

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.
Propositional Resolution
Complete strategy

Result of the algorithm

- $\Theta_k =$ set of minimum deduction clauses.
- $\Gamma$ and $\Theta_k$ are equivalent.

Property 2.2.5
For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Property 2.2.6
The sets $\Gamma$ and $\Theta_k$ are equivalent.

Proof.

$$\Gamma \equiv \Delta_0 \cup \emptyset = \Delta_0 \cup \Theta_0 \equiv \ldots \equiv \Delta_k \cup \Theta_k = \emptyset \cup \Theta_k = \Theta_k$$
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

- Introduced by Martin Davis and Hilary Putnam in 1960, then refined by Martin Davis, George Logemann and Donald Loveland in 1962
- Indicates if a set of clauses is satisfiable and exhibits a model.
- Basis for (most efficient) complete SAT-solvers such as chaff, zchaff and satz.
Principle I

Two types of formulae transformations:

1. preserving the truth value:
   - reduction
Principle I

Two types of formulae transformations:

1. preserving the truth value:
   - reduction

2. preserving only satisfiability:
   - pure literal elimination
   - unit resolution

DPLL is efficient because it uses these two kinds transformations.
Principle II

“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to `true` a heuristically chosen variable (branching literal).
- **Continue the algorithm recursively.**
**Principle II**

“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to **true** a heuristically chosen variable (branching literal).
- Continue the algorithm recursively.
- **Backtracking**: If we arrive to a contradiction, we return to the last choice, and we “branch” by assigning **false** to the chosen variable.
The DPLL Algorithm (figure 2.1)

**bool function** `Algo_DPLL(Γ: set of clauses)`

0. Remove the valid clauses from $\Gamma$.
   - If $\Gamma = \emptyset$, return `true`.
   - Else return `DPLL(Γ)`

**bool function** `DPLL(Γ: set of non-valid clauses)`
The function returns true if and only if $\Gamma$ is satisfiable.

1. If $\bot \in \Gamma$, return `false`.
   - If $\Gamma = \emptyset$, return `true`.
2. Reduce $\Gamma$.
3. Remove from $\Gamma$ the clauses containing a pure literal.
   - If the set $\Gamma$ has been modified, goto 1.
4. Apply unit resolution to $\Gamma$.
   - If the set $\Gamma$ has been modified, goto 1.
5. Pick an arbitrary variable $x$ in $\Gamma$
   return `DPLL(Γ[x := 0])` or else `DPLL(Γ[x := 1])`
The DPLL Algorithm (figure 2.1)

**bool function** `Algo_DPLL( \Gamma: set of clauses)`
0. **Remove the valid clauses** from \( \Gamma \).
   - If \( \Gamma = \emptyset \), return (\text{true}).
   - Else return (\text{DPLL}(\Gamma))

**bool function** `DPLL( \Gamma: set of non-valid clauses)`
The function returns true if and only if \( \Gamma \) is satisfiable.
1. If \( \bot \in \Gamma \), return (\text{false}).
   - If \( \Gamma = \emptyset \), return (\text{true}).
2. **Reduce** \( \Gamma \).
3. **Remove from** \( \Gamma \) **the clauses containing a pure literal**.
   - If the set \( \Gamma \) has been modified, goto 1.
4. **Apply unit resolution to** \( \Gamma \).
   - If the set \( \Gamma \) has been modified, goto 1.
5. Pick an arbitrary variable \( x \) in \( \Gamma \)
   return (\text{DPLL}(\Gamma[x := 0])) or else \text{DPLL}(\Gamma[x := 1]))
The DPLL Algorithm (figure 2.1)

\textbf{bool function} \textsc{Algo\_DPLL}( \Gamma: \text{set of clauses})
0 \quad \text{Remove the valid clauses from} \ \Gamma.
   \quad \text{If} \ \Gamma = \emptyset, \text{return} (\text{true}).
   \quad \text{Else} \ \text{return} (\textsc{DPLL}(\Gamma))

\textbf{bool function} \textsc{DPLL}( \Gamma: \text{set of non-valid clauses})
\text{The function returns true if and only if} \ \Gamma \text{is satisfiable.}
1 \quad \text{If} \ \bot \in \Gamma, \text{return} (\text{false}).
   \quad \text{If} \ \Gamma = \emptyset, \text{return} (\text{true}).
2 \quad \text{Reduce} \ \Gamma.
3 \quad \text{Remove from} \ \Gamma \ \text{the clauses containing a pure literal.}
   \quad \text{If} \ \text{the set} \ \Gamma \ \text{has been modified, goto} \ 1.
4 \quad \text{Apply unit resolution to} \ \Gamma.
   \quad \text{If} \ \text{the set} \ \Gamma \ \text{has been modified, goto} \ 1.
5 \quad \text{Pick an arbitrary variable} \ x \ \text{in} \ \Gamma
\quad \text{return} (\textsc{DPLL}(\Gamma[x := 0])) \ \text{or else} \ \textsc{DPLL}(\Gamma[x := 1]))
The DPLL Algorithm (figure 2.1)

bool function Algo_DPLL( Γ: set of clauses)
0 Remove the valid clauses from Γ.
   If Γ = ∅, return (true).
   Else return (DPLL(Γ))

bool function DPLL( Γ: set of non-valid clauses)
The function returns true if and only if Γ is satisfiable.
1 If ⊥ ∈ Γ, return(false).
   If Γ = ∅, return (true).
2 Reduce Γ.
3 Remove from Γ the clauses containing a pure literal.
   If the set Γ has been modified, goto 1.
4 Apply unit resolution to Γ.
   If the set Γ has been modified, goto 1.
5 Pick an arbitrary variable x in Γ
return (DPLL(Γ[x := 0]) or else DPLL(Γ[x := 1]))
Removal of clauses containing a pure literal

Definition 2.3.1
A literal $L$ is **pure** if none of the clauses in $\Gamma$ contains $L^c$.

Lemma 2.3.2
Removing clauses with a pure literal preserves satisfiability.

Proof: see exercise 48.

Intuition: assigning $[L]_v$ to 1 is always possible for a pure literal.
Example 2.3.3

Let $\Gamma$ be the set of clauses

1. $p + q + r$
2. $\neg q + \neg r$
3. $q + s$
4. $\neg s + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure. Therefore we obtain

2. $q + r$
3. $q + s$

The literals $r$ and $s$ are pure. We obtain the empty set. Therefore $\Gamma$ has a model (for instance $p = 1$, $t = 1$, $r = 0$, $s = 1$).
Example 2.3.3

Let $\Gamma$ be the set of clauses

(1) $p + q + r$
(2) $\overline{q} + \overline{r}$
(3) $q + s$
(4) $\overline{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure. Therefore we obtain

(2) $\overline{q} + \overline{r}$
(3) $q + s$
Example 2.3.3

Let $\Gamma$ be the set of clauses

(1) $p + q + r$
(2) $\bar{q} + \bar{r}$
(3) $q + s$
(4) $\bar{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure.
Therefore we obtain

(2) $\bar{q} + \bar{r}$
(3) $q + s$

The literals $\bar{r}$ and $s$ are pure.
Example 2.3.3

Let $\Gamma$ be the set of clauses

1. $p + q + r$
2. $\overline{q} + \overline{r}$
3. $q + s$
4. $\overline{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure. Therefore we obtain

2. $\overline{q} + \overline{r}$
3. $q + s$

The literals $\overline{r}$ and $s$ are pure. We obtain the empty set.
Example 2.3.3

Let $\Gamma$ be the set of clauses

1. $p + q + r$
2. $\bar{q} + \bar{r}$
3. $q + s$
4. $\bar{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure.
Therefore we obtain

2. $\bar{q} + \bar{r}$
3. $q + s$

The literals $\bar{r}$ and $s$ are pure.
We obtain the empty set.
Therefore $\Gamma$ has a model (for instance $p = 1$, $t = 1$, $r = 0$, $s = 1$).
Propositional Resolution
The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Unit resolution

Definition 2.3.4
A unit clause is a clause which contains only one literal.
Unit resolution

**Definition 2.3.4**

A unit clause is a clause which contains only one literal.

**Lemma 2.3.5**

Let $L$ be the literal from a unit clause of $\Gamma$.
Let $\Theta$ be the set of clauses obtained by:

- removing the clauses containing $L$
- removing $L^c$ inside the remaining clauses

- if $\Gamma$ contains two complementary unit clauses, then $\Theta = \{\bot\}$.

We apply this process for every unit clause.
$\Gamma$ has a model if and only if $\Theta$ has a model.

Proof: The proof is requested in exercise 49.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. $\Gamma = p + q, \bar{p}, \bar{q}$

Hence $\Gamma$ has no model.

2. $\Gamma = a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{b}, d, \bar{c}$

Then $\Gamma$ has no model.

3. $\Gamma = p, q, p + r, \bar{p} + r, q + \bar{r}, \bar{q} + s$

By unit resolution, we obtain: $r, s$.

This set of clauses has a model, hence $\Gamma$ has a model.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

- \( \Gamma = p + q, \overline{p}, \overline{q} \)
  - \( q, \overline{q} \) by unit resolution on \( \overline{p} \), then \( \bot \) by UR on \( \overline{q} \)
  - Hence \( \Gamma \) has no model.

- \( \Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c} \)
  - \( a, a \)
  - \( \bot \) hence \( \Gamma \) has no model.

- \( \Gamma = p, q, p + r, \overline{p} + r, q + \overline{r}, \overline{q} + s \)
  - By unit resolution, we obtain: \( r, s \)
  - This set of clauses has a model, hence \( \Gamma \) has a model.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \bar{p}, \bar{q} \)
   - by unit resolution on \( \bar{p} \), then \( \bot \) by UR on \( \bar{q} \)
   - Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{b}, d, \bar{c} \)
   - By unit resolution, we obtain: \( r, s \).
   - This set of clauses has a model, hence \( \Gamma \) has a model.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \overline{p}, \overline{q} \)

   \( q, \overline{q} \) by unit resolution on \( \overline{p} \), then \( \bot \) by UR on \( \overline{q} \)

   Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c} \)

   1. \( a, \overline{a} \).
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \bar{p}, \bar{q} \)
   - \( q, \bar{q} \) by unit resolution on \( \bar{p} \), then \( \bot \) by UR on \( \bar{q} \)
   - Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{b}, d, \bar{c} \)
   - 1. \( a, \bar{a} \).
   - 2. \( \bot \)
   - hence \( \Gamma \) has no model.

3. \( \Gamma = p, q, p + r, \bar{p} + r, q + \bar{r}, \bar{q} + s \)
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

- $\Gamma = p + q, \overline{p}, \overline{q}$
  
  \begin{itemize}
  \item $q, \overline{q}$ by unit resolution on $\overline{p}$, then $\bot$ by UR on $\overline{q}$
  \end{itemize}

  Hence $\Gamma$ has no model.

- $\Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c}$

  \begin{enumerate}
  \item $a, \overline{a}$.
  \item $\bot$
  \end{enumerate}

  hence $\Gamma$ has no model.

- $\Gamma = p, q, p + r, \overline{p} + r, q + \overline{r}, \overline{q} + s$

  By unit resolution, we obtain: $r, s$. 
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \bar{p}, \bar{q} \)
   - \( q, \bar{q} \) by unit resolution on \( \bar{p} \), then \( \bot \) by UR on \( \bar{q} \)
   - Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{b}, d, \bar{c} \)
   - 1. \( a, \bar{a} \)
   - 2. \( \bot \)
   - Hence \( \Gamma \) has no model.

3. \( \Gamma = p, q, p + r, \bar{p} + r, q + \bar{r}, \bar{q} + s \)
   - By unit resolution, we obtain: \( r, s \).
   - This set of clauses has a model, hence \( \Gamma \) has a model.
Removal of valid clauses

Lemma 2.3.7

Let $\Theta$ be the set of clauses obtained by removing the valid clauses of $\Gamma$. $\Gamma$ has a model iff $\Theta$ has a model.

Proof.

$\Rightarrow$ Every model of $\Gamma$ is clearly a model of $\Theta$, since $\Theta \subseteq \Gamma$. 
Removal of valid clauses

Lemma 2.3.7

Let $\Theta$ be the set of clauses obtained by removing the valid clauses of $\Gamma$.

$\Gamma$ has a model iff $\Theta$ has a model.

Proof.

$\Rightarrow$ Every model of $\Gamma$ is clearly a model of $\Theta$, since $\Theta \subseteq \Gamma$.

$\Leftarrow$ Suppose that $\Theta$ has a model $\nu$.

Let $\nu'$ be the truth assignment built from $\nu$ by assigning any value to the variables appearing in $\Gamma$ but not in $\Theta$.

Every clause $C$ in $\Gamma$ is:

- either a clause of $\Theta$, then $[C]_{\nu'} = [C]_{\nu} = 1$
- or a valid clause, so obviously $\nu'$ is a model of $C$.

Hence $\nu'$ is a model of $\Gamma$. 

The DPLL Algorithm (figure 2.1)

bool function Algo_DPLL( Γ: set of clauses)
0  Remove the valid clauses from Γ.
   If Γ = ∅, return (true).
   Else return (DPLL(Γ))

bool function DPLL( Γ: set of non-valid clauses)
The function returns true if and only if Γ is satisfiable.
1  If ⊥ ∈ Γ, return (false).
   If Γ = ∅, return (true).
2  Reduce Γ.
3  Remove from Γ the clauses containing a pure literal.
   If the set Γ has been modified, goto 1.
4  Apply unit resolution to Γ.
   If the set Γ has been modified, goto 1.
5  Pick an arbitrary variable x in Γ
   return (DPLL(Γ[x := 0]) or else DPLL(Γ[x := 1]))
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$. 

Since every leave contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\neg a + \neg b$, $a + b$, $\neg a + c$, $a + c$, $\neg b + c$, $b + c$. 

\[
\neg a + \neg b, a + b, \neg a + c, a + c, \neg b + c, b + c
\]
Example 2.3.8

Let \( \Gamma \) be the set of clauses: \( \overline{a} + \overline{b}, a + b, \overline{a} + \overline{c}, a + c, \overline{b} + \overline{c}, b + c \).
Example 2.3.8

Let \( \Gamma \) be the set of clauses: \( \overline{a} + \overline{b}, a + b, \overline{a} + \overline{c}, a + c, \overline{b} + \overline{c}, b + c \).
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$. 

\[
\begin{align*}
\overline{a} + \overline{b}, a + b, \overline{a} + \overline{c}, a + c, \overline{b} + \overline{c}, b + c \\
\text{RED} \\
\text{UR} \\
\bot
\end{align*}
\]
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$.

Since every leaf contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + b$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$.

\[
\begin{array}{c}
\overline{a} + b, \ a + b, \overline{a} + \overline{c}, \ a + c, \overline{b} + \overline{c}, b + c \\
a=0 \\
b, c, \overline{b} + \overline{c}, b + c
\end{array}
\]

\[
\begin{array}{c}
\overline{a} + b, \ a + b, \overline{a} + \overline{c}, \ a + c, \overline{b} + \overline{c}, b + c \\
a=1 \\
\overline{b}, \overline{c}, \overline{b} + \overline{c}, b + c
\end{array}
\]

Since every leave contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$.

Since every leave contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$.

Since every leaf contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\bar{p} + q, \bar{p} + s, p + q, \bar{p} + \bar{s}$. 

Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{p} + q, p + s, p + q, \overline{p} + \overline{s}$.

Since one branch leads to the empty set, the set $\Gamma$ is satisfiable. It is **useless** to continue the construction of the right branch.
Theorem 2.3.9 et 2.3.10

The algorithm Algo\_DPLL is correct and terminates.
Theorem 2.3.9 et 2.3.10

The algorithm Algo-DPLL is correct and terminates.

Termination proof

- Valid clause removal is only executed once
- Simplification iteration: the number of clauses strictly decreases
- Recursive calls: the number of variables strictly decreases

Hence the termination.
Correctness proof

- Invariant for the simplification loop:
  
  the current value of $\Gamma$ has a model iff $\Gamma$ has a model.
Correctness proof

- Invariant for the simplification loop:
  the current value of $\Gamma$ has a model iff $\Gamma$ has a model.
  see lemma for each simplification.
Correctness proof

▶ Invariant for the simplification loop:

the current value of $\Gamma$ has a model iff $\Gamma$ has a model.

see lemma for each simplification.

▶ Correctness of recursive calls:

Reminder of property 2.1.21:

$\Gamma$ has a model iff $\Gamma[x := 0]$ or $\Gamma[x := 1]$ is satisfiable.

So if the recursive calls are correct, the current call is too.
Correctness proof

- Invariant for the simplification loop:
  the current value of $\Gamma$ has a model iff $\Gamma$ has a model.
  See lemma for each simplification.

- Correctness of recursive calls:
  *Reminder of property 2.1.21:*
  $\Gamma$ has a model iff $\Gamma[x := 0]$ or $\Gamma[x := 1]$ is satisfiable.
  So if the recursive calls are correct, the current call is too.

Since the algorithm is correct for a set $\Gamma$ with no literal, it is correct for any set $\Gamma$ of clauses.
Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).
Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).

- **Choice of the variable (branching literal):**
  - A good choice for variable $x$ in step (5) is the variable that appears most often.
  - A better choice is the variable which will lead to the maximum number of simplifications

Cf. Sub-section 2.3.5, for the main branching heuristics
Overview

Correctness

Completeness

Introduction to resolution algorithms

Complete strategy

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Resolution is a correct and complete deductive system: it characterizes all the unsatisfiable formulae.

Complete Strategy is an algorithm for computing every clause deducible from an initial set.

The DPLL algorithm uses ideas from resolution to:
  - find a model
  - or else, prove the unsatisfiability by an efficient search of the assignments.
Next lecture

- Natural deduction

Homework: **Hypotheses** :

- (H1) : $p \Rightarrow \neg j \equiv \neg p \lor \neg j$
- (H2) : $\neg p \Rightarrow j \equiv p \lor j$
- (H3) : $j \Rightarrow m \equiv \neg j \lor m$
- ($\neg C$): $\neg m \land \neg p$ (two clauses)

**Build** the proofs of $H1, H2, H3, \neg C \vdash \bot$ obtained by:

- Complete Strategy
- DPLL (you may pick any variable for branching)