Organization

12 weeks:

- Lecture, 1h30 / week
- Seminar 2 \( \times 1h30 = 3h / week \)

This week

- Lecture : today ! and Thursday at 15h15 as usual
- Seminar : 1 session on Wednesday this week, 2 starting next week

see http://inf242.forge.imag.fr/
## Planning

<table>
<thead>
<tr>
<th>Important dates</th>
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<tbody>
<tr>
<td><strong>Winter break:</strong> February 20th - 26th</td>
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<tr>
<td><strong>Project pre-report:</strong> February 28th or March 1st</td>
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<td><strong>Midterm exam:</strong> March 6th - 10th</td>
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<td><strong>Spring break:</strong> April 17th - 23rd</td>
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<td><strong>Project report:</strong> April 25th or 26th</td>
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<td><strong>Project defense:</strong> May 2nd - 5th</td>
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<td><strong>Final exam:</strong> May 9th - 19th</td>
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<td><strong>Second session:</strong> Week starting June 19th</td>
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Final mark

Evaluations

- Assessments 60%:
  4 periodic tests 10%, midterm exam 20% and project 30%
- Exam: 40%

Project groups: 3-4 students per project group.

- Part 1: Modeling of a logic problem (automated in a software)
- Part 2: Transforming instances of these problems in clauses and solving them using an SAT solver

Examples of problems: Takuzu, Squaro, Sudoku, Master Mind ...
Course Material

- Lectures handout (in French, with holes)
- Subject of the project (on the website)
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Summary

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Logic

Definitions

- Logic is used to specify what a correct reasoning is, regardless of the application domain.
Logic

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- **Logic** is used to specify what a correct reasoning is, regardless of the application domain.
- A **reasoning** is a way to obtain a conclusion starting from given hypotheses.
Logic

Definitions

- **Logic** is used to specify what a correct reasoning is, regardless of the application domain.
- A **reasoning** is a way to obtain a conclusion starting from given hypotheses.
- A **correct** reasoning does not say anything about the truth of the hypotheses, it only says that *starting from the truth of the hypotheses, one can deduct the truth of the conclusion.*
Examples

Example I

- **Hypothesis I**: All men are mortal
- **Hypothesis II**: Socrates is a man
- **Conclusion**: Socrates is mortal
Examples

Example I

- **Hypothesis I:** All men are mortal
- **Hypothesis II:** Socrates is a man
- **Conclusion:** Socrates is mortal

Example II

- **Hypothesis I:** All that is rare is expensive
- **Hypothesis II:** A cheap horse is rare
- **Conclusion:** A cheap horse is expensive!
Adding a hypothesis

Hypothesis I: All that is rare is expensive

Hypothesis II: A cheap horse is rare

Hypothesis III: Every cheap thing is "not expensive"

Conclusion: Contradictory hypotheses! Since:

Hypothesis I + Hypothesis II: A cheap horse is expensive

Hypothesis III: A cheap horse is not expensive
Adding a hypothesis

Example III

- **Hypothesis I**: All that is rare is expensive
- **Hypothesis II**: A cheap horse is rare
- **Hypothesis III**: Every cheap thing is “not expensive”
Adding a hypothesis

### Example III

- **Hypothesis I:** All that is rare is expensive
- **Hypothesis II:** A cheap horse is rare
- **Hypothesis III:** Every cheap thing is “not expensive”
- **Conclusion:** Contradictory hypotheses! Since:
  - **Hypothesis I + Hypothesis II:** A cheap horse is expensive
  - **Hypothesis III:** A cheap horse is not expensive
Some history…

- **George Boole** (1815-1864)
  - *symbolic logic*: first try at reasoning without natural language

- **Gottlob Frege** (1848-1925)
  - *propositional calculus*: formal rules for reasoning
  - *proof theory*: a proof itself becomes a mathematical object

- **Bertrand Russell** (1872-1970)
  - *logicism*: attempt at a formalization of all existing mathematics
  - *paradox* found in earlier systems

- **Kurt Gödel** (1906-1978)
  - *completeness* of the first-order predicate calculus
  - *incompleteness theorem* for systems including arithmetic

- **Alonzo Church** (1903-1995)
  - *lambda-calculus*: a proof is an algorithm and vice-versa
Applications

- **Hardware**: logic gates
- **Software verification and correctness**: 
  - Tools: provers COQ, PVS, Prover9, MACE, ...
  - **Meteor** (ligne 14)
- **Artificial Intelligence**: 
  - expert system (**MyCin**), ontology
- **Programming**: **Prolog**
  - artificial intelligence
  - natural language processing
- **Mathematical proofs, Security, ...**
Overview of the Semester

TODAY

▶ Propositional logic
▶ Propositional resolution
▶ Natural deduction for propositional logic

MIDTERM EXAM

▶ First order logic
▶ Logical basis for automated proving
  (“first-order resolution”)  
▶ First-order natural deduction

EXAM
Course Objectives

- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
Course Objectives

▶ Modeling and formalizing a problem.
▶ Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
▶ Reasoning, that is, building a correct reasoning using the tools of propositional logic and first order logic.
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- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
- Reasoning, that is, building a correct reasoning using the tools of propositional logic and first order logic.
- Writing a rigorous proof, in particular an induction.
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Propositional Logic

Definition

Propositional logic is the logic *without quantifiers.*

The only logical operations used are:

- $\neg$ (negation),
- $\land$ (conjunction, also known as logical “and”),
- $\lor$ (disjunction, also known as logical “or”),
- $\Rightarrow$ (implication),
- $\Leftrightarrow$ (equivalence)
Example: **Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

**Conclusion** (C): Mary is the sister of John, or Peter is old.
Example: Formal reasoning

Hypotheses:

- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

Conclusion (C): Mary is the sister of John, or Peter is old.

- p: ”Peter is old”
- j: ”John is the son of Peter”
- m: ”Mary is the sister of John”
**Example: Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

**Conclusion (C):** Mary is the sister of John, or Peter is old.

- \( p \): "Peter is old"
- \( j \): "John is the son of Peter"
- \( m \): "Mary is the sister of John"

\[
\begin{align*}
\text{(H1)}: & \quad p \Rightarrow \neg j \\
\text{(H2)}: & \quad \neg p \Rightarrow j \\
\text{(H3)}: & \quad j \Rightarrow m \\
\text{(C)}: & \quad m \lor p
\end{align*}
\]
**Example: Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
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**Conclusion (C):** Mary is the sister of John, or Peter is old.

- \( p \): "Peter is old"
- \( j \): "John is the son of Peter"
- \( m \): "Mary is the sister of John"

We prove that \( H1 \land H2 \land H3 \Rightarrow C \):

\[
(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p
\]

is true regardless of the truth value of propositions \( p, j, m \).
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Vocabulary of the language

- The constants: $\top$ (true) and $\bot$ (false)
- The variables: for example, $x$, $y_1$
- The parentheses
- The connectives: $\neg$, $\lor$, $\land$, $\Rightarrow$, $\Leftrightarrow$
(Strict) Formula

Definition 1.1.1

A strict formula is defined inductively as:

- $\top$ and $\bot$ are strict formulae.
- A variable is a strict formula.
- If $A$ is a strict formula then $\neg A$ is a strict formula.
- If $A$ and $B$ are strict formulae and if $\circ$ is one of the following operations $\lor, \land, \Rightarrow, \Leftrightarrow$ then $(A \circ B)$ is a strict formula.
(Strict) Formula

Definition 1.1.1

A strict formula is defined inductively as:

- \( \top \) and \( \bot \) are strict formulae.
- A variable is a strict formula.
- If \( A \) is a strict formula then \( \neg A \) is a strict formula.
- If \( A \) and \( B \) are strict formulae and if \( \circ \) is one of the following operations \( \lor, \land, \Rightarrow, \Leftrightarrow \) then \( (A \circ B) \) is a strict formula.

Example 1.1.2

\((a \lor (\neg b \land c))\) is a strict formula, but not \(a \lor (\neg b \land c)\), nor \((a \lor (\neg (b) \land c))\).
Example 1.1.3

The structure of the formula \( (a \lor (\neg b \land c)) \) is illustrated by the following tree:
Example 1.1.3

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The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:
Exercise

\(((p \land \neg(p \lor q)) \land \neg r)\)
Exercise

\[ ((p \land \neg(p \lor q)) \land \neg r) \]
### Size of a formula

#### Definition 1.1.10

The **size of a formula** $A$, denoted $|A|$, is inductively defined as:

- $|\top| = 0$ and $|\bot| = 0$.
- If $A$ is a variable then $|A| = 0$.
- $|\neg A| = 1 + |A|$.
- $|(A \circ B)| = |A| + |B| + 1$.

---

Example 1.1.11

$|\left(a \lor \left(\neg b \land c\right)\right)| = 3$. 

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B. Wack (UGA)
Size of a formula

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Example 1.1.11

$|(a \lor (\neg b \land c))| =$
Size of a formula

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Example 1.1.11

$|(a \lor (\neg b \land c))| = 3$. 
First result

Strict formulae decompose uniquely in their sub-formulae.

Theorem 1.1.13

For every formula $A$, there is one and only one of the following cases:

- $A$ is a variable,
- $A$ is a constant,
- $A$ can be written in a unique manner as $\neg B$ where $B$ is a formula,
- $A$ can be written in a unique manner as $(B \circ C)$ where $B$ and $C$ are formulae.

This will allow us to:

- prove properties by cases
- perform structural induction on the formulae rather than induction on their size.
Prioritized formula

Definition 1.1.14

A prioritized formula is inductively defined in a similar way but:

▶ if $A$ and $B$ are prioritized formulae the $A \circ B$ is a prioritized formula,

▶ if $A$ is a prioritized formula then $(A)$ is a prioritized formula.

Example 1.1.15

$a \lor \neg b \land c$ is a prioritized formula, but not a (strict) formula.
Connective precedence

**Definition 1.1.16**

By decreasing precedence, the connectives are: \( \lnot, \land, \lor, \Rightarrow \) and \( \Leftrightarrow \).

**Left associativity**

For identical connectives, the left-hand side connective has higher precedence:

\[
A \circ B \circ C = (A \circ B) \circ C
\]

*except for the implication:* \( A \Rightarrow B \Rightarrow C = A \Rightarrow (B \Rightarrow C) \)
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of
- $a \land b \lor c$ is the abbreviation of
- $a \lor b \land c$ is the abbreviation of
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of $(a \land b) \land c$
- $a \land b \lor c$ is the abbreviation of $(a \land b) \lor c$
- $a \lor b \land c$ is the abbreviation of $a \lor (b \land c)$
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Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. 

\([A]_v\) denotes the truth value of the formula \(A\) for the assignment \(v\).
Truth assignment of a formula

Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \( \{0, 1\} \).

\([A]_\nu\) denotes the truth value of the formula \( A \) for the assignment \( \nu \).

Example: Let \( \nu \) be an assignment such that \( \nu(x) = 0 \) and \( \nu(y) = 1 \).
Applying \( \nu \) to \( x \lor y \) is written as
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A truth assignment is a function from the set of variables of a formula to the set \( \{0, 1\} \).

\([A]_v\) denotes the truth value of the formula \( A \) for the assignment \( v \).

**Example:** Let \( v \) be an assignment such that \( v(x) = 0 \) and \( v(y) = 1 \).

Applying \( v \) to \( x \lor y \) is written as \([x \lor y]_v\).

\([x \lor y]_v = \)
Truth assignment of a formula

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\([x \lor y]_\nu = 0 \lor 1 = 1\)

Conclusion:
Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. 

\([A]_v\) denotes the truth value of the formula \(A\) for the assignment \(v\).

Example: Let \(v\) be an assignment such that \(v(x) = 0\) and \(v(y) = 1\). Applying \(v\) to \(x \lor y\) is written as \([x \lor y]_v\)

\([x \lor y]_v = 0 \lor 1 = 1\)

Conclusion: \(x \lor y\) is true for the truth assignment \(v\)
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $\nu$ a truth assignment.

- $[x]_\nu = \nu(x)$
- $[\top]_\nu = 1$, $[\bot]_\nu = 0$
- $[\neg A]_\nu = 1 - [A]_\nu$
- $[(A \lor B)]_\nu = \max\{[A]_\nu, [B]_\nu\}$
- $[(A \land B)]_\nu = \min\{[A]_\nu, [B]_\nu\}$
- $[(A \Rightarrow B)]_\nu = \begin{cases} 1 & \text{if } [A]_\nu = 0 \\ [B]_\nu & \text{otherwise} \end{cases}$
- $[(A \Leftrightarrow B)]_\nu = \begin{cases} 1 & \text{if } [A]_\nu = [B]_\nu \\ 0 & \text{otherwise} \end{cases}$
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- $[(A \rightarrow B)]_\nu =$
- $[(A \leftrightarrow B)]_\nu =$
Truth value of a formula

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3. $[\neg A]_\nu =
4. [A \lor B]_\nu = \max\{[A]_\nu, [B]_\nu\}
5. [A \land B]_\nu = \min\{[A]_\nu, [B]_\nu\}
6. [A \Rightarrow B]_\nu = \text{if } [A]_\nu = 0 \text{ then } 1 \text{ else } [B]_\nu
7. [A \Leftrightarrow B]_\nu = \text{if } [A]_\nu = [B]_\nu \text{ then } 1 \text{ else } 0
Truth value of a formula

Definition 1.2.2
Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

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- $[\neg A]_v =$
- $[(A \lor B)]_v =$
- $[(A \land B)]_v =$
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Truth value of a formula

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- $[(A \Rightarrow B)]_v = $ 
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- $[(A \leftrightarrow B)]_\nu = $
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- $[(A \Leftrightarrow B)]_v = \text{if } [A]_v = [B]_v \text{ then } 1 \text{ else } 0$
Definition 1.2.3

A truth table of a formula $A$ is a table representing the truth values of $A$ for all the possible values of the variables of $A$.

- a line of the truth table = an assignment
- a column of the truth table = the truth values of a formula.
Basic tables

0 indicates false and 1 indicates true.
The value of the constant $\top$ is 1 and the value of the constant $\bot$ is 0

Table 1.1 (truth table of connectives)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\neg x$</th>
<th>$x \lor y$</th>
<th>$x \land y$</th>
<th>$x \Rightarrow y$</th>
<th>$x \Leftrightarrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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</tr>
</tbody>
</table>
Example:

**Example 1.2.4**

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x \implies y</th>
<th>\neg x</th>
<th>\neg x \lor y</th>
<th>(x \implies y) \iff (\neg x \lor y)</th>
<th>x \lor \neg y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ⇒ y</th>
<th>¬x</th>
<th>¬x ∨ y</th>
<th>(x ⇒ y) ⇔ (¬x ∨ y)</th>
<th>x ∨ ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>¬x</td>
<td>x ∨ ¬y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<td></td>
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<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ⇒ y</th>
<th>¬x</th>
<th>¬x ∨ y</th>
<th>(x ⇒ y) ⇔ (¬x ∨ y)</th>
<th>x ∨ ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example:

**Example 1.2.4**

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \Rightarrow y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \iff (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \Rightarrow y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \Leftrightarrow (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
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<td>$1$</td>
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<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Equivalent formulae

**Definition 1.2.5**

Two formulae $A$ and $B$ are equivalent (denoted $A \equiv B$ or simply $A = B$) if they have the same truth value for every assignment.
Equivalent formulae

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Example 1.2.6

$$x \Rightarrow y \equiv \neg x \lor y$$
Equivalent formulae

**Definition 1.2.5**

Two formulae $A$ and $B$ are equivalent (denoted $A \equiv B$ or simply $A = B$) if they have the same truth value for every assignment.

**Example 1.2.6**

$x \Rightarrow y \equiv \neg x \lor y$

Remark:
The logical connective $\iff$ does not mean $A \equiv B$. 
Validity, tautology (1/2)

Definition 1.2.8

- A formula is valid if its value is 1 for all truth assignments.
- A valid formula is also called a tautology.
- Denoted by $\models A$. 

Example 1.2.9

- $(x \Rightarrow y) \iff (\neg x \vee y)$ is valid;
- $x \Rightarrow y$ is not valid since it is false for $x = 1$ and $y = 0$. 

Definition 1.2.8

- A formula is **valid** if its value is 1 for all truth assignments.
- A valid formula is also called a **tautology**.
- Denoted by \( \models A \).

Example 1.2.9

- \((x \Rightarrow y) \iff (\neg x \lor y)\) is valid;
- \(x \Rightarrow y\) is not valid since it is false for \(x = 1\) and \(y = 0\).
Valid, tautology (2/2)

Property 1.2.10

The formulae $A$ and $B$ are equivalent ($A \equiv B$) if and only if

formula $A \Leftrightarrow B$ is valid.

Proof.

The property is a consequence of the truth table of $\Leftrightarrow$. □
Model for a formula

Definition 1.2.11

A truth assignment \( v \) for which a formula has truth value equal to 1 is a model for that formula.

\( v \) satisfies \( A \) or \( v \) makes \( A \) true.

Example 1.2.12

A model for \( x \Rightarrow y \) is:
Model for a formula

Definition 1.2.11

A truth assignment $\nu$ for which a formula has truth value equal to 1 is a model for that formula.

$\nu$ satisfies $A$ or $\nu$ makes $A$ true.

Example 1.2.12

A model for $x \Rightarrow y$ is:

$x = 1, y = 1$ (among others)
Model for a formula

Definition 1.2.11

A truth assignment \( \nu \) for which a formula has truth value equal to 1 is a model for that formula.

\( \nu \) satisfies \( A \) or \( \nu \) makes \( A \) true.

Example 1.2.12

A model for \( x \Rightarrow y \) is:

\[ x = 1, \ y = 1 \] (among others)

On the opposite, \( x = 1, \ y = 0 \) is not a model for \( x \Rightarrow y \).
Model for a set of formulae

Definition 1.2.13
\[ \nu \text{ is a model for a set of formulae } \{A_1, \ldots, A_n\} \]
if and only if
it is a model for every formula in the set.
Model for a set of formulae

Definition 1.2.13

\( \nu \) is a model for a set of formulae \( \{A_1, \ldots, A_n\} \)
if and only if
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Example 1.2.14

A model of \( \{a \implies b, b \implies c\} \) is:
Model for a set of formulae

Definition 1.2.13

\( \nu \) is a model for a set of formulae \( \{A_1, \ldots, A_n\} \) if and only if it is a model for every formula in the set.

Example 1.2.14

A model of \( \{a \Rightarrow b, b \Rightarrow c\} \) is:

\[
a = 0, b = 0 \text{ (for any } c) .
\]
Property of a model for a set of formulae

Property 1.2.15

$\nu$ is a model for \( \{ A_1, \ldots, A_n \} \)
if and only if
$\nu$ is a model for \( A_1 \land \ldots \land A_n \).
Property of a model for a set of formulae

Property 1.2.15

\( \nu \) is a model for \( \{ A_1, \ldots, A_n \} \)

if and only if

\( \nu \) is a model for \( A_1 \land \ldots \land A_n \).

Example 1.2.16

The set of formulae \( \{ a \Rightarrow b, b \Rightarrow c \} \)

and the formula \( (a \Rightarrow b) \land (b \Rightarrow c) \)

have identical models.
A truth assignment $v$ which yields the value 0 for a formula is a counter-model for the formula.

$v$ does not satisfy the formula or $v$ makes the formula false.
Counter-model

**Definition 1.2.17**

A truth assignment \( \nu \) which yields the value 0 for a formula is a **counter-model** for the formula.

\( \nu \) does not satisfy the formula or \( \nu \) makes the formula **false**.

**Example 1.2.18**

A counter-model of \( x \Rightarrow y \) is:
Counter-model

Definition 1.2.17

A truth assignment $v$ which yields the value 0 for a formula is a counter-model for the formula.

$v$ does not satisfy the formula or $v$ makes the formula false.

Example 1.2.18

A counter-model of $x \Rightarrow y$ is:

$x = 1, y = 0.$
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is **satisfiable** if it admits a model.

Definition 1.2.21
A (set of) formula(e) is **unsatisfiable** if it is not satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is **satisfiable** if it admits a model.

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A (set of) formula(e) is **unsatisfiable** if it is not satisfiable.

Example 1.2.22
\( x \land \neg x \) is unsatisfiable, but \( x \implies y \) is satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is **satisfiable** if it admits a model.

Definition 1.2.21
A (set of) formula(e) is **unsatisfiable** if it is not satisfiable.

Example 1.2.22

\[ x \land \neg x \text{ is unsatisfiable, but } x \Rightarrow y \text{ is satisfiable.} \]

Beware
unsatisfiable = 0 model
invalid = at least 1 counter-model

satisfiable = at least 1 model
valid = 0 counter-model
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Today

- Why define and use formal logic?
- Propositional logic:
  - 1 variable = 1 proposition (a statement) which may be true or false
  - 5 connectives to articulate these propositions
- Meaning of formulae:
  - assignment = choice of a truth value for each variable
  - a formula may be true for 0, 1, several or every assignment
Next time

Homework: build the truth table for the “Peter, John and Mary” example.

► Important equivalences
► Substitutions and replacements
► Normal Forms
Oxford’s motto

The more I study, the more I know
The more I know, the more I forget
The more I forget, the less I know